XIII. *On the Motion of Two Spheres in a Fluid.*

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The general theory of the motion of a single rigid body through an infinite incompressible fluid is well known, chiefly through the work of Thomson and Tait* and Kirchhoff,† and we are able to calculate numerically the results in the case of the sphere, the ellipsoid, and a large number of cylindrical surfaces. The theory of the motion of two or more bodies in a fluid has naturally not made the same progress, and we are unable to determine the form of the expressions involved for the general motion of any particular solids. So far as I am aware, the first attempt was made by Stokes, in a paper read before the Cambridge Philosophical Society in 1843, entitled "On some cases of Fluid Motion."‡ In this paper, amongst other problems, he considers the case of two spheres. He determines the instantaneous velocity potential for two concentric spheres and for two concentric cylinders with fluid between them, and finds that the effect of the fluid is to increase the inertia of the inner sphere by a mass $=\frac{1}{2}.\frac{a^3+2b^3}{b^3-a^3}$ of the mass of the fluid displaced, and that of the inner cylinder by a mass $\frac{b^2+a^2}{b^2-a^2}$ of the mass displaced, a, b, being the radii of the spheres or cylinders. He also approximates to the cases where one sphere is moving in the presence of another in an infinite fluid; and also in the presence of a plane, the method used being first to calculate the velocity potential for any motion of the points of the plane, and then suppose them actually animated with velocities equal and opposite to the normal velocities of the fluid motion at those points if the plane had been removed. He applies the same method also to the consideration of the motion of two spheres. In a note in the Report of the British Association at Oxford, 1847, he states the theorem given by me in § 4 relating to the image of a doublet whose axis passes through the centre, and mentions that this will easily serve to determine the motion. In 1863 Herr Bjerknes communicated a paper to the Society of Sciences at Christiana, on the motion of a sphere which changes its volume, and in

* Nat. Phil., p. 264, new edition, p. 330.

† Borchardt, Bd. 71.

‡ Camb. Phil. Trans., vol. viii.

which he approximates for the motion of two spheres. I have not been able to see this paper, nor some others which he presented to the same Society at some later periods; but he has given an account of his researches in the 'Comptes Rendus,'* together with some historical notices on the development of the theory. He does not seem, however, to have been acquainted with the important paper of STOKES above referred to.† In 1867 THOMSON and TAIT'S 'Natural Philosophy' appeared, containing general theorems on the motion of a sphere in a fluid bounded by an infinite plane, viz.: that a sphere moving perpendicularly to the plane moves as if repelled by it, whilst if it moves parallel to it it is attracted. In a paper on vortex motion in the same year (Edin. Trans., vol. xxv.), THOMSON proved that a body or system of bodies passing on one side of a fixed obstacle move as if attracted or repelled by it, according as the translation is in the direction of the resultant impulse or opposite to it. In the 'Philosophical Magazine' for June, 1871, Professor GUTHRIE publishes some letters from Sir W. THOMSON on the apparent attraction or repulsion between two spheres, one of which is vibrating in the line of centres. Results only are given, and he states that if the density of the free globe is less than that of the fluid, there is a "critical" distance beyond which it is attracted, and within which it is repelled. The problem of two small spheres is also considered by KIRCHHOFF in his 'Vorles. ü. Math. Phys.,' pp. 229, 248. In his later papers BJERKNES takes up the question of "pulsations" as well as vibrations. Of solutions for other cases than spheres, KIRCHHOFF has considered‡ the case of two thin rigid rings, the axes of the rings being any closed

* 'Comptes Rendus,' tom. lxxxiv., p. 1222, &c.

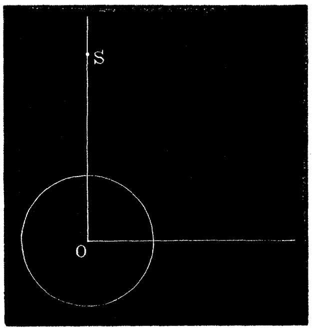
† Not only Herr BJERKNES, but several writers on the Continent seem to be unacquainted with this paper of STOKES, and also with GREEN'S papers. KIRCHHOFF, in his 'Vorlesungen über Mathematische Physik' (second edition, p. 227), says that DIRICHLET first treated the motion of a sphere in a fluid in the Monatsberichte der Berl. Akad.' in 1852, and CLEBSCH that of the ellipsoid in 1856, in 'Crelle,' Bd. 52. BJERKNES also repeats the same statement, and CLEBSCH in his paper regards DIRICHLET as the first to solve for the sphere. In his paper DIRICHLET says: "Wie es scheint, ist bis jetzt für keinen noch so einfachen Fall der Widerstand, den ein in einer ruhenden Flüssigkeit fortbewegter fester Körper von dieser erleidet, aus den seit Euler bekannten allgemeinen gleichungen der Hydrodynamik abgeleitet worden." The fact is that GREEN in a paper read before the Royal Society of Edinburgh in 1833, entitled "Researches on the Vibrations of Pendulums in Fluid Media" (Trans. Roy. Soc. Edin.; also published in the Reprint of his papers, p. 313), and written without the knowledge of POISSON'S paper of 1831, "Sur les mouvements simultanés d'un pendule et de l'air environnant," treated of the motions of an ellipsoid moving parallel to one of its axes. He obtains the velocity potential as an elliptic integral for a motion parallel to an axis, which also of course contains implicitly that for the sphere. He shows that it is necessary to suppose the density of the body augmented by a quantity proportional to the density of the fluid. For the case of the spheroids moving in their equatorial planes or parallel to their axes he completely determines this quantity, whilst for the sphere he finds that it is one-half the mass of the fluid displaced. The first place in which I have been able to find the well known form of the velocity potential for a sphere is in STOKES' paper of 1843 before mentioned. He obtains it as a particular case of a more general problem, and refers to it as the "known" value for the sphere. The equations of the lines of flow were, I believe, first given by DIRICHLET.

‡ BORCHARDT, Bd. 71.

curves and the sections by planes perpendicular to the axis being small circles of constant radii, and he arrives at the result that their action on one another may be represented by supposing electric currents to flow round them; and I have recently solved the problem of the motion of two cylinders in any manner with their axes always parallel. The velocity potentials for the motion of the two cylinders are found in general as definite integrals, which, when the cylinders move as a rigid body, are expressed in a simple finite form as elliptic functions of bipolar coordinates. The functions involved in the coefficients of the velocities in the expression for the energy have a close analogy with those for spheres arrived at in the following investigation.

1. Our first aim will be to find the velocity potential for the motion of the fluid in which a sphere is fixed and in which a source of fluid exists. By the image of the source in general is meant that collocation of sources or sinks within the sphere which produces outside of it a fluid motion which in conjunction with the original source has no normal motion across the sphere: in other words, that "mass" of positive or negative sources which produces across the surface of the sphere a normal flow equal and opposite to that of the outside source. When this "image" is found, the way is *theoretically* clear to finding the velocity potential when two spheres are fixed in the fluid, and thence, by distributing over the surface of the spheres sources proportional to the normal motion of the surface at that point, to determine the velocity potential when the two spheres are moving in any manner. In the case of an electrical point the image is, as is well known, a negative point at the inverse point of the other. In the case of fluid motion the image is, as will be shown, a positive source at the inverse point, together with a negative line sink stretching from this point to the centre of the sphere.

Fig. 1.



2. Take O the centre of the sphere for origin and let the axis of z pass through the source S. Let the radius of the sphere be a, and the distance of S from the centre be b. Then the velocity potential will clearly be symmetrical about O S. The velocity potential for the unit source at S can be expanded in the series

$$-\frac{1}{R}=-\frac{1}{\sqrt{r^2-2br\cos\theta+b^2}}=-\frac{1}{b}-\Sigma_1^\infty\frac{r^n}{b^{n+1}}P_n$$

which holds good for points where $r<b$, whence when $r=a(<b)$ the flow *into* the sphere at any point (θ) is

$$\Sigma_1^\infty \frac{na^{n-1}}{b^{n+1}} P_n$$

Expand the potential due to the sources, &c., inside the spheres in a series of spherical harmonics

$$V=\Sigma_0^\infty \frac{a^n}{r^{n+1}} Y_n (r>a)$$

Hence the flow *out* of the sphere, for points just outside, is

$$-\Sigma_0^\infty (n+1)\frac{1}{a^2} Y_n$$

and this must be equal to the other, whence

$$Y_n = -\frac{n}{n+1}\left(\frac{a}{b}\right)^{n+1} P_n \text{ and } Y_0=0$$

and

$$V=-\Sigma_1^\infty \frac{n}{n+1}\frac{a^{2n+1}}{(br)^{n+1}} P_n$$

$$=-\Sigma_1^\infty \frac{a^{2n+1}}{(br)^{n+1}} P_n + \Sigma_1^\infty \frac{1}{n+1}\frac{a^{2n+1}}{(br)^{n+1}} P_n$$

Consider

$$\chi=\frac{\mu'}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}=\mu'\Sigma_0^\infty \frac{\lambda^n}{r^{n+1}} P_n \begin{pmatrix}\lambda<a\\ r>a\end{pmatrix}$$

the potential for a source μ' at a point on O S inside the sphere at a distance λ from the centre. Then

$$\int_0^\lambda \left(\chi-\frac{\mu'}{r}\right) d\lambda=\mu'\Sigma_1^\infty \frac{1}{n+1}\frac{\lambda^{n+1}}{r^{n+1}} P_n$$

Comparing this with the expression for V, we see that if we make $\lambda=\frac{a^2}{b}$ and $\mu'=\frac{a}{b}\times$ source

$$V=-\frac{a}{b}\Sigma_1^\infty \frac{\lambda^n}{r^{n+1}} P_n+\frac{1}{\lambda}\int_0^\lambda \chi d\lambda-\frac{\mu'}{r}$$

$$=-\frac{a}{b}\frac{1}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}+\frac{1}{a}\int_0^\lambda \frac{d\lambda}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}$$

i.e., V is the potential of a source at the distance $\frac{a^2}{b}$ from O whose magnitude is equal to $\frac{a}{b}$ of the source at S, together with a line sink extending from O to the distance $\frac{a^2}{b}$, the line density of the sink being $\frac{1}{a}\times$ source at S.

Performing the integration for V, we find finally that the whole velocity potential for a unit source at S is

$$\phi=-\frac{1}{\mathrm{SP}}+\mathrm{V}=-\frac{1}{\sqrt{r^2-2br\cos\theta+b^2}}-\frac{a}{b}\frac{1}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}$$
$$+\frac{1}{a}\log\frac{\lambda-r\cos\theta+\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}{r(1-\cos\theta)}$$

where $\lambda\equiv\frac{a^2}{b}$

It is easy to verify this value for ϕ by direct differentiation.

If we apply the same method to find the velocity potential for the motion of fluid inside a sphere under the influence of a source inside, the integral becomes infinite unless the source is zero. The case is of course physically impossible since if fluid is generated within the sphere it must pass through the boundary. But if we also place an equal sink at any point within, the motion is then possible, and the expression becomes finite. S being the source let S′ be its inverse point with reference to the sphere, and S″ any point on the line S S′ produced to infinity. Then the "image" of S is a source $\frac{\mu a}{b}$ at S′, and a line distribution of sinks of line density $\frac{\mu}{a}$ from S′ to infinity. Let S_1 be an equal sink, then its image and that of S will produce potentials with finite derivatives. In fact, the potential at P will be

$$\phi=\mu\left\{\frac{1}{\mathrm{SP}}-\frac{1}{\mathrm{S_1P}}+\frac{a}{b}\cdot\frac{1}{\mathrm{S'P}}-\frac{a}{b_{\prime}}\cdot\frac{1}{\mathrm{S'_1P}}-\frac{1}{a}\log\frac{\mathrm{OS'}-r\cos\theta+\mathrm{S'P}}{\mathrm{OS'_1}-r\cos\theta_{\prime}+\mathrm{S'_1P}}\cdot\frac{1-\cos\theta_{\prime}}{1-\cos\theta}\right\}$$

where θ, $\theta_{\prime}$ are the angles O P makes respectively with O S, O S_1.

3. The expression found for the motion when there is a single source outside the sphere enables us to deduce the velocity potential for a single sphere moving through an infinite fluid. Taking the direction of motion as the axis of x, from which we will suppose θ measured, we may arrange a surface distribution of sources proportional to $\cos\theta d\mathrm{S}$ and integrate over the surface of the sphere, or we may employ the simpler method used in a paper in the 'Quarterly Journal of Mathematics' for March, p. 128. The first gives us the velocity potential when the sphere moves by an integration which would be laborious. The other gives directly the potential, when the sphere is fixed and the fluid moves past it, by means of an easy differentiation. Putting a source at $x=b$ and an equal sink at $x=-b$, let these move off to infinity, increasing indefinitely as they do so, yet so that the motion at a finite distance from the origin is finite. In the limit we clearly get the case of fluid flowing past the sphere.

We have to find the limit when $b=\infty$ $\frac{\mu}{b^2}=k$ of

$$\phi=-\mu\left\{\frac{1}{\sqrt{r^2-2br\cos\theta+b^2}}-\frac{1}{\sqrt{r^2+2br\cos\theta+b^2}}\right.$$
$$+\frac{a}{b}\left(\frac{1}{\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}-\frac{1}{\sqrt{r^2+2\lambda r\cos\theta+\lambda^2}}\right)$$
$$\left.-\frac{1}{a}\log\frac{\lambda-r\cos\theta+\sqrt{r^2-2\lambda r\cos\theta+\lambda^2}}{\lambda+r\cos\theta+\sqrt{r^2+2\lambda r\cos\theta+\lambda^2}}\cdot\frac{1+\cos\theta}{1-\cos\theta}\right\}$$

When b is large and λ small this is easily shown to be

$$\phi=-\frac{\mu}{b^2}\left\{2r\cos\theta+\frac{a^3\cos\theta}{r^2}+\frac{A}{b}+\ldots\right\}$$

Hence the limit is

$$\phi=-k\left(2x+\frac{a^3x}{r^3}\right)$$

If the velocity of the fluid at an infinite distance parallel to x is u towards the origin, then

$$2k=u$$

Also impressing on the whole system a velocity u, the sphere moves with velocity u in an infinite fluid, and the potential function is

$$\phi=-\frac{a^3ux}{2r^3}=-\frac{a^3u\cos\theta}{2r^2}$$

The well-known form of ϕ in this case.

4. If now two spheres A, B are present in the fluid, and we consider the series of images resulting from the first image in A, we see that they very rapidly become extremely complicated, *e.g.*, the first image is a source and line sink; the image of this in B consists of (1) a source and line sink, (2) the image of the first line sink or a line sink (segment of a circle), and an area source bounded by this last line sink and two straight lines from the centre. It is, therefore, hopeless in this way to find first the velocity potential for a source in the presence of the two spheres, and thence the potential for any motion of the spheres. But now suppose A fixed and B moving in any direction. If A were not present the velocity potential of B would be that due to a doublet at its centre, whose axis lies in the direction of the motion of B. The effect of the introduction of A will be to produce a series of images of this doublet, lying inside A and B. This method dispenses with the necessity of integrating over the spheres when we have found the velocity potential for the doublet. In the special

case where B is moving in the line joining the centres, the image becomes simplified and reduces to a single doublet. For let us find the image of a doublet whose axis passes through the centre of a sphere.

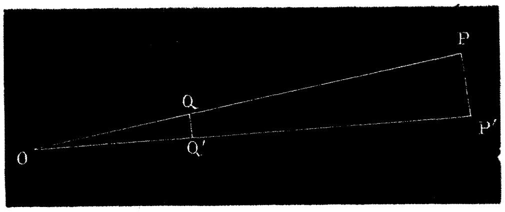
The doublet is formed by allowing an equal source and sink P, P′ to indefinitely approach one another, their magnitudes increasing indefinitely, yet so that $\mu.\overline{PP'}$ is finite. Now let P, P′ lie on the line through the centre of the sphere, and let Q, Q′ be their inverse points; moreover, let the limit of $\mu.\overline{PP'}=k$. Then the image of P, P′ consists of a source $\frac{\mu a}{OP}$ at Q, a sink $\frac{\mu a}{OP'}$ at Q′, and a line source (supposing P outside P′, and therefore Q′ outside Q) along Q Q′ with line density $\frac{\mu}{a}$, also the quantity $\frac{\mu}{a}.QQ'$, together with the sink at Q′, is equal and opposite to the source at Q, and we may suppose it added to the sink at Q′, when they become equal. Now as P, P′ approach to coincidence so do Q, Q′, and the image of the doublet k at P becomes the doublet at Q, whose magnitude is the limit of

$$\frac{\mu a}{OP}.QQ'=k.\frac{a}{OP}.\frac{QQ'}{PP'}=-k.\frac{a^3}{OP^3},$$

i.e., one of opposite sign and magnitude $\left(\frac{a}{OP}\right)^3\times$ that at P. The same result can easily be shown to follow from the analytical formula in § 2.

The case where the doublet has its axis perpendicular to the line joining the centres has more analogy with the case of a source. The image here consists of a doublet of the *same* sign at the inverse point, with a trail of doublets of opposite sign extending to the centre.

Fig. 2.



Let, as before, P, P′ be equal source and sink, Q, Q′ their inverse points with respect to the circle.

Then at Q, Q′ we have a source and sink of magnitude $\frac{\mu a}{OP}$, and in the limit we have a doublet

$$\frac{\mu a\overline{QQ'}}{OP}=k\left(\frac{a}{OP}\right)^3$$

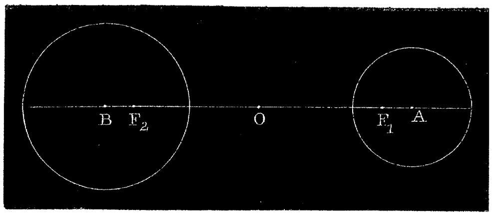
Also, if R, R′ be corresponding points on O Q, O Q′, we have a line density $-\frac{\mu}{a}$ at

R and $+\frac{\mu}{a}$ at R′. Consequently when P, P′ approach indefinitely so do R, R′, and we get a line doublet along O Q, whose line magnitude at any point R is the limit of

$$-\frac{\mu}{a}.\mathrm{RR}'=-\frac{\mu}{a}\cdot\frac{\mathrm{OR}}{\mathrm{OP}}.\mathrm{PP}'=-\frac{k}{a}\cdot\frac{\mathrm{OR}}{\mathrm{OP}}$$

i.e., proportional to the distance from the centre.

Fig. 3.



5. Supposing that the positions of all the images of the doublets and their magnitudes are known when the sphere A is moving along the line B A, we proceed to find an expression for the kinetic energy. Let ρ_n be the distance of the n^{th} image in A from A, and σ_n the distance of the n^{th} image in B. Also let the magnitudes of the doublets there be μ_n, ν_n respectively. Let ϕ be the velocity potential of the motion, and ϕ_n, ϕ'_n the parts of ϕ due to μ_n and ν_n. Then denoting the kinetic energy by T

$$2\mathrm{T}=-\int[\phi]\frac{\delta\phi}{\delta n}d\mathrm{S}=-2\pi a^2u\int_0^\pi[\phi]\sin\theta\cos\theta d\theta$$

where $[\phi]$ is the value of ϕ at any point $(a.\theta)$ on the sphere. Now $\phi=\Sigma\phi_n+\Sigma\phi'_n$ and the part of T due to ϕ_n will be

$$2\mathrm{T}=-2\pi a^2u\int_0^\pi\frac{\mu_n(a\cos\theta+\rho_n)\sin\theta\cos\theta}{\{a^2+2\rho_n a\cos\theta+\rho_n^{\,2}\}^{\frac{3}{2}}}d\theta$$

$$=-2\pi a^2\mu_n u\int_{-1}^{+1}\frac{(\rho_n+a\mu)\mu d\mu}{(a^2+\rho^2_n+2\rho_n a\mu)^{\frac{3}{2}}}$$

Now

$$\int_{-1}^{+1}\frac{(\rho+a\mu)\mu d\mu}{\{a^2+\rho^2+2\rho a\mu\}^{\frac{3}{2}}}=-\frac{d}{d\rho}\int_{-1}^{+1}\frac{\mu d\mu}{\sqrt{a^2+\rho^2+2\rho a\mu}}$$

$$=\frac{d}{d\rho}\cdot\frac{1}{3\rho^2a^2}\{(\rho+a)(\rho^2+a^2-\rho a)-(\rho-a)(\rho^2+a^2+\rho a)\}$$

When $\rho\equiv\rho_n$ $\rho_n<a$ and the above becomes

$$\frac{d}{d\rho}\cdot\frac{2\rho}{3a^2}=\frac{2}{3a^2}$$

Similarly when $\rho \equiv \sigma_n$ $\sigma_n > a$ and it becomes

$$\frac{d}{d\rho}\cdot\frac{2a}{3\rho^2}=-\frac{4a}{3\rho^3}$$

and

$$2\mathrm{T}=-\tfrac{4}{3}\pi.u\Sigma_0^\infty \mu_n+\tfrac{8}{3}\pi a^3 u\Sigma_1^\infty\left(\frac{\nu_n}{\sigma_n{}^3}\right)$$

Also μ the original doublet $=-\frac{a^3u}{2}$, and if M_1 be the mass of fluid displaced by the sphere A

$$2\mathrm{T}=\tfrac{1}{2}\mathrm{M}_1u^2\left\{1+\Sigma_1^\infty\frac{\mu_n}{\mu}\right\}+\mathrm{M}_1u^2\Sigma_1^\infty\left(-\frac{\nu_n a^3}{\mu\sigma_n{}^3}\right)$$

But from what has been shown before

$$\mu_n=-\frac{a^3}{\sigma_n{}^3}\nu_n$$

Hence

$$2\mathrm{T}=\tfrac{1}{2}\mathrm{M}_1u^2\left\{1+3\Sigma_1^\infty\left(\frac{\mu_n}{\mu}\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By § 4 we have, if c is the distance between the centres,

$$\mu_n=-\frac{a^3}{\sigma_n{}^3}\nu_n$$

$$\nu_n=-\left(\frac{b}{c-\rho_{n-1}}\right)^3\mu_{n-1}$$

Also

$$\rho_n=\frac{a^2}{\sigma_n} \qquad c-\sigma_n=\frac{b^2}{c-\rho_{n-1}}$$

Hence

$$\mu_n=\left(\frac{ab}{\sigma_n(c-\rho_{n-1})}\right)^3\mu_{n-1}=\left(\frac{b\rho_n}{a(c-\rho_{n-1})}\right)^3\mu_{n-1}$$

$$=\left(\frac{b}{a}\right)^{3n}\left\{\frac{\rho_n\rho_{n-1}\ldots\ldots\rho_1}{(c-\rho_{n-1})\ldots\ldots(c-\rho_1)c}\right\}^3\mu$$

Again

$$\rho_n=\frac{a^2}{\sigma_n}=\frac{a^2}{c-\dfrac{b^2}{c-\rho_{n-1}}}$$

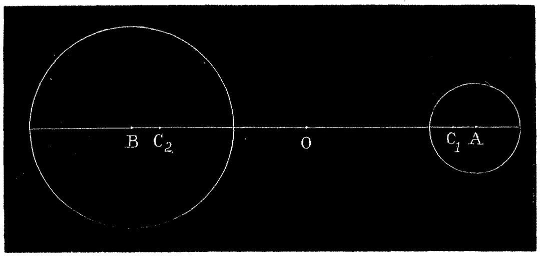
whence

$$\rho_n\rho_{n-1}-\frac{c^2-b^2}{c}\rho_n-\frac{a^2}{c}\rho_{n-1}+a^2=0$$

Put $\rho_n = u_n + x$, and choose x so as to make the constant term vanish. To find x we have

$$x^2 - \frac{a^2+c^2-b^2}{c}x + a^2 = 0$$

Fig. 4.



Now let C_1 C_2 be the inverse points of the spheres, and O the middle point of C_1 C_2. Put $C_1C_2 = 2\lambda$. Then

$$OA = \sqrt{\lambda^2 + a^2} = \frac{a^2+c^2-b^2}{2c} = r_1 \text{ say} \quad . \quad . \quad . \quad . \quad (2)$$

$$\lambda = \sqrt{r_1^2 - a^2} \quad . \quad . \quad . \quad . \quad (3)$$

$$c = r_1 + r_2$$

Further, P being any point on the sphere A, denote the constant ratio $\frac{C_2P}{C_1P}$ by q_1 and let q_2 be the similar constant for the sphere B. Then

$$\left.\begin{aligned} q_1 &= \frac{\lambda + r_1 - a}{\lambda - r_1 + a} = \frac{\lambda + r_1}{a} = \frac{a}{r_1 - \lambda} \\ q_2 &= \frac{\lambda - r_2 + b}{\lambda + r_2 - b} = \frac{b}{\lambda + r_2} = \frac{r_2 - \lambda}{b} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

The equation to determine x now becomes

$$x^2 - 2r_1x + a^2 = 0$$

The roots of which are $r_1 \pm \lambda$.

Choosing the positive sign, the equation of differences becomes

$$u_n u_{n-1} - \left(x_2 - \frac{a^2}{c}\right)u_n + \left(x_1 - \frac{a^2}{c}\right)u_{n-1} = 0$$

Now $a^2 = x_1x_2$ whence writing $\frac{1}{v_n}$ for u_n we get

$$v_n - \frac{x_2(c-x_1)}{x_1(c-x_2)} v_{n-1} = -\frac{c}{x_1(c-x_2)}$$

Here

$$\frac{c-x_2}{c-x_1}=\frac{r_2+\lambda}{r_2-\lambda}=q_2^{-2}$$

and

$$\frac{x_1}{x_2}=\frac{r_1+\lambda}{r_1-\lambda}=q_1^{2}$$

$$\frac{c}{x_1(c-x_2)}=\frac{c}{(r_1+\lambda)(r_2+\lambda)}$$

Whence writing $\frac{q_2}{q_1}\equiv q$

$$(\mathrm{E}-q^2)v_n=-\frac{c}{(r_1+\lambda)(r_2+\lambda)}$$

and

$$v_n=\mathrm{A}q^{2n}-\frac{1}{2\lambda}$$

and

$$\rho_n=aq_1+\frac{1}{-\frac{1}{2\lambda}+\mathrm{A}q^{2n}}$$

But

$$n=0\ \rho=0$$

$$\therefore \mathrm{A}=\frac{1}{2\lambda}-\frac{1}{\lambda+r_1}=\frac{r_1-\lambda}{2\lambda(r_1+\lambda)}=\frac{1}{2\lambda q_1^2}$$

and

$$\rho_n=aq_1-\frac{2\lambda}{1-q_1^{-2}q^{2n}}$$

$$=(r_1-\lambda)\frac{1-q^{2n}}{1-q_1^{-2}q^{2n}}$$

Also

$$c-\rho_n=r_1+r_2-(r_1+\lambda)+\frac{2\lambda}{1-q_1^{-2}q^{2n}}$$

$$=(r_2+\lambda)\frac{1-q^{2n+2}}{1-q_1^{-2}q^{2n}}$$

$$\therefore \frac{b}{a}\cdot\frac{\rho_n}{c-\rho_{n-1}}=q\cdot\frac{1-q_1^{-2}q^{2n-2}}{1-q_1^{-2}q^{2n}}=q\frac{p_{n-1}}{p_n}\text{ say}$$

and

$$\mu_n=\left\{q^n.\frac{p_{n-1}}{p_n}.\frac{p_{n-2}}{p_{n-1}}\cdot\cdot\cdot\cdot\frac{p_0}{p_1}\right\}^3\mu$$

$$=\left\{\frac{(1-q_1^{-2})q^n}{1-q_1^{-2}q^{2n}}\right\}^3\mu$$

Whence

$$2T=\tfrac{1}{2}M_1u^2\left\{1+3(1-q_1^{-2})^3\Sigma_1^\infty\left(\frac{q^n}{1-q_1^{-2}q^{2n}}\right)^3\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We shall denote, in what follows,

$$(1-q_1^{-2})^3\Sigma_1^\infty\left(\frac{q^n}{1-q_1^{-2}q^{2n}}\right)^3$$

by the functional symbol $Q\left(\frac{1}{q_1}, q\right)$.

6. If the sphere B is also moving along the line AB, the kinetic energy of the fluid will be of the form

$$2T=\tfrac{1}{2}M_1u_1^2\left\{1+3Q\left(\frac{1}{q_1},q\right)\right\}+\tfrac{1}{2}M_2u_2^2\{1+3Q(q_2,q)\}+Lu_1u_2$$

It remains, then, to find the value of L.

It is easily seen that L depends on the part of ϕ belonging to the images of B's motion taken over the sphere A, together with that belonging to the images of A's motion taken over the sphere B. Let now dashed letters apply to the images, &c., of the B system, then using the results in § 5, the part of L due to the integration over A is

$$=-\tfrac{4}{3}\pi u_1\Sigma_1^\infty\mu'_n+\tfrac{8}{3}\pi a^3u_1\Sigma_0^\infty\left(\frac{\nu'_n}{\sigma'^3_n}\right)$$

But as before, remembering that now the original doublet is in B,

$$\mu'_n=-\left(\frac{a}{\sigma'_{n-1}}\right)^3\nu'_{n-1}$$

$$\mu'_1=-\left(\frac{a}{c}\right)^3\nu'=\left(\frac{a}{c}\right)^3\cdot\frac{b^3u_2}{2}$$

and

$$L_1=-\tfrac{4}{3}\pi u_1\Sigma_1^\infty(3\mu'_n)$$

$$=-2\pi u_1u_2\frac{a^3b^3}{c^3}\Sigma_1^\infty\left(\frac{\mu'_n}{\mu'_1}\right)$$

and

$$\mu'_n=\left\{\frac{b\rho'_n}{a(c-\rho'_{n-1})}\right\}^3\mu'_{n-1}$$

$$=\left(\frac{b}{a}\right)^{3n-3}\left\{\frac{\rho'_n.\rho'_{n-1}\ldots\ldots\rho'_2}{(c-\rho'_{n-1})\ldots\ldots(c-\rho'_1)}\right\}^3\mu'_1$$

Now as before

$$\rho'_n=aq_1+\frac{1}{-\frac{1}{2\lambda}+Aq^{2n}}$$

and determining A by the condition that $\rho_1'=\frac{a^2}{c}$ we shall find

$$\rho'_n=aq_1-\frac{2\lambda}{1-q^{2n}}=(r_1-\lambda)\frac{1-q_1^2q^{2n}}{1-q^{2n}}$$

and

$$c-\rho'_{n-1}=(r_2+\lambda)\frac{1-q_1^2q^{2n}}{1-q^{2n-2}}$$

$$\frac{b}{a}\frac{\rho'_n}{c-\rho'_{n-1}}=q\cdot\frac{1-q^{2n-2}}{1-q^{2n}}$$

$$\mu'_n=\left\{\frac{(1-q^2)q^{n-1}}{1-q^{2n}}\right\}^3\mu'_1$$

and

$$\mathrm{L}_1=-2\pi u_1u_2\Sigma\left\{\frac{ab(1-q^2)}{c}\cdot\frac{q^{n-1}}{1-q^{2n}}\right\}^3$$

$$=-16\pi u_1u_2\lambda^3\Sigma_1\left(\frac{q^n}{1-q^{2n}}\right)^3$$

Similarly $\mathrm{L}_2=$ same quantity.

Therefore, denoting by M′ the mass of fluid contained in a sphere of radius unity

$$\mathrm{L}=-4\pi u_1u_2\mathrm{Q}'(q)=-3\mathrm{M}'u_1u_2\mathrm{Q}'(q)$$

where

$$\mathrm{Q}'(q)=\Sigma_1^\infty\left(\frac{2\lambda q^n}{1-q^{2n}}\right)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Tables for Q and Q′ are given at the end of the paper for equal spheres, and for the case of $a=2b$.

7. When the sphere A is moving perpendicularly to B A, the original doublet is one perpendicular to the line B A, as also its images. Suppose A is moving along the axis of x, A B being the axis of z. Then the normal velocity at a point P on the sphere A is $v\sin\theta\cos\chi$, $(\alpha.\theta.\chi)$ being the polar coordinates of P; and the kinetic energy is given by

$$2\mathrm{T}=-a^2v\int_0^\pi\int_0^{2\pi}[\phi]\sin^2\theta\cos\chi\,d\theta d\chi$$

Let μ be the magnitude of a doublet at a point distant ρ from the centre of A; the part of ϕ depending on this is

$$\frac{\mu r\sin\theta\cos\chi}{\{r^2+\rho^2+2\rho r\cos\theta\}^{\frac{3}{2}}}$$

and the part of 2T depending on this is

$$-\mu a^3 v\int_0^\pi\int_0^{2\pi}\frac{\sin^3\theta\cos^2\chi d\theta d\chi}{\{a^2+\rho^2+2\rho a\cos\theta\}^{\frac{3}{2}}}$$

$$=-\mu\pi a^3 v\int_0^\pi\frac{\sin^3\theta d\theta}{\{a^2+\rho^2+2\rho a\cos\theta\}^{\frac{3}{2}}}$$

The integral of which is

$$-\frac{2\mu\pi v}{3\rho^3}[(\rho^2+a^2)\{(\rho+a)-(\rho\backsim a)\}-\rho a\{\rho+a+(\rho\backsim a)\}]$$

Writing ν and σ for μ, ρ for doublets outside the sphere A, we obtain

$$-\tfrac{4}{3}\pi\mu v \text{ and } -\frac{4\pi a^3\nu v}{3\sigma^3}$$

whence

$$2\text{T}=-\text{M}_1 v\Sigma\left\{\frac{\mu}{a^3}+\frac{\nu}{\sigma^3}\right\}$$

Now any ν at the distance σ produces an image in A consisting of a doublet $\nu\left(\frac{a}{\sigma}\right)^3$ at a distance $\frac{a^2}{\sigma}$, together with a line sink stretching from this to the centre, whose line magnitude is $-\frac{\nu}{a\sigma}\times$ distance from the centre. Hence the whole amount of the image is

$$\nu\left(\frac{a}{\sigma}\right)^3-\frac{1}{2}\frac{\nu}{a\sigma}\left(\frac{a^2}{\sigma}\right)^2=\tfrac{1}{2}\nu\left(\frac{a}{\sigma}\right)^2$$

Now every μ except μ_0 forms part of an image of some ν, and of that ν only. Hence

$$\Sigma\frac{\nu}{\sigma_3}=2\Sigma\frac{\mu}{a^3}-2\frac{\mu_0}{a^3}$$

and

$$2\text{T}=-\frac{\text{M}_1 v}{a^3}\{\mu_0+3\Sigma\mu\}$$

$$=\tfrac{1}{2}\text{M}_1 v^2\left\{1+3\Sigma\left(\frac{\mu}{\mu_0}\right)\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The Σ extending to the whole mass of *images* inside A.

8. If A has also a motion along B A, together with one perpendicular to it, T has no term depending on u, v; for it is clear that if the sign of v is changed, then the kinetic energy must be the same as before.

If B moves also perpendicularly to B A, T will have additional terms in v_2^2, and v_1, v_2. The coefficient of v_2^2 will be analogous to that for v_1^2, whilst that for v_1, v_2, as in the case of u_1, u_2, consists of two parts, depending on the integration over the two spheres. As in the case of L this coefficient L′ is

$$L'v_1v_2 = -\tfrac{4}{3}\pi v_1\Sigma\left\{\mu'+\nu'\left(\frac{a}{\sigma'}\right)^3\right\}-\tfrac{4}{3}\pi v_2\Sigma\left\{\nu+\mu\left(\frac{b}{\sigma}\right)^3\right\}$$

dashed letters referring to the motion of B, μ, ν referring to images within A and B respectively, and σ' σ denoting distances from the centres of A and B. This may be reduced as in the former case to

$$L'v_1v_2 = -4\pi v_1\Sigma_1(\mu')-4\pi v_2\Sigma_1(\nu)$$

the μ' being the images in A of B's motion perpendicular to A B, and ν the images in B of A's motion perpendicular to A B.

$$L'v_1v_2 = -4\pi v_1\nu_0\Sigma_1\left(\frac{\mu'}{\nu_0}\right)-4\pi v_2\mu_0\Sigma\left(\frac{\nu}{\mu_0}\right)$$

where ν_0, μ_0 are the original doublets at the centres of B, A, *i.e.*

$$\nu_0 = -\frac{b^3v_2}{2} \quad \mu_0 = -\frac{a^3v_1}{2}$$

whence

$$L' = \tfrac{3}{2}M_2\Sigma_1\left(\frac{\mu'}{\nu_0}\right)+\tfrac{3}{2}M_1\Sigma\left(\frac{\nu}{\mu_0}\right) \quad \ldots\ldots\ldots \quad (8)$$

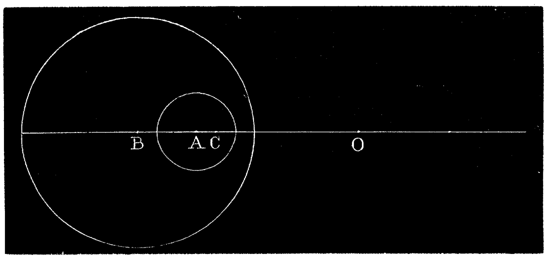
in which last the ratios $\frac{\mu'}{\nu_0}$, $\frac{\nu}{\mu_0}$ do not contain v_1, v_2.

In the case of general motion of two spheres, each will have three components of velocity, u_1, v_1, w_1; u_2, v_2, w_2; and, in general, the expression for the kinetic energy will contain 21 terms. In the case in question we can easily see that the coefficients of 12 of these vanish. For consider the term in u_1, w_2—suppose v_1, w_1, u_2, v_2 all zero; the energy, from the symmetry of the motion, must clearly be unaltered if we reverse the direction of w_2. And this can only happen if the coefficient of $u_1w_2=0$. In this way we find the terms all vanish except those in u_1^2, v_1^2, w_1^2, u_2^2, v_2^2, w_2^2, $u_1\,u_2$, $v_1\,v_2$, $w_1\,w_2$. Also from symmetry the coefficients of v_1^2, v_2^2, $v_1\,v_2$, are equal respectively to the coefficients of w_1^2, w_2^2, $w_1\,w_2$.

In what has gone before we have expressed the coefficient of u_1^2, u_2^2, $u_1\,u_2$ in terms of quantities determined by the radii and distance of the spheres, and have shown how the coefficients of the other terms depend on the images of the motion, whereby we can without much difficulty approximate to their values when the distance of the spheres is large compared with their radii—or the distance between their surfaces is large compared with the radius of one of the spheres. We pass on to consider the case where one sphere is inside the other. An approximation to the value of the coefficients of v_1^2 and $v_1 v_2$ is given in § 15. It is remarkable that in the case of two cylinders the coefficients of the terms in u^2, v^2 are equal, while those of $u_1 u_2$ and $v_1 v_2$ are equal and opposite. But this is due to the fact that in a cylinder the image of a doublet (or a source) is a single doublet, whatever be the direction of the axis of the original doublet.

9. If S S_1 be in a line through the centre the infinite trail of images of § 2 cuts out, and we are left with an image source and sink, and a line sink between them, supposing S to be outside S_1. Let now S and S_1 approach together and become a doublet whose strength is μ. Then we shall get a single doublet as its image whose strength $=\frac{\mu a}{b}\mathrm{L}.\frac{S'S'_1}{SS_1}=-\frac{\mu a^3}{b^3}$ as in the former case. This we might have deduced at once from the case of the external doublet in § 4, considered as the image of its image.

Fig. 5.



If we proceed to find the kinetic energy, as in the previous case, we must clearly be led to the same *form* for the result, viz.: when A is moving with a velocity u from B

$$2T=\tfrac{1}{2}M_1u^2\left\{1+3\Sigma_1^\infty\left(\frac{\mu_n}{\mu}\right)\right\}$$

where μ_n . . . . are the strengths of the doublets inside A alone. But in this case the relations between the μ, ρ, σ are given by the equations (a, b being the radii of sphere)

$$\mu_n=-\frac{a^3}{\sigma_n^{\;3}}\nu_n,$$

$$\nu_n=-\left(\frac{b}{c+\rho_{n-1}}\right)^3\mu_{n-1}$$

$$\rho_n=\frac{a^2}{\sigma_n}\qquad c+\sigma_n=\frac{b^2}{c+\rho_{n-1}}$$

whence

$$\rho_n=\frac{a^2}{\dfrac{b^2}{c+\rho_{n-1}}-c}$$

$$\rho_n\rho_{n-1}+\frac{c^2-b^2}{c}\rho_n+\frac{a^2}{c}\rho_{n-1}+a^2=0$$

which differs from the equation for external spheres in having $-\rho$ for ρ for all values of n. We may therefore use the same solution and writing here

$$\mathrm{OA}=\sqrt{\lambda^2+a^2}$$

$$\sqrt{\lambda^2+b^2}-\sqrt{\lambda^2+a^2}=c$$

$$\mathrm{OA}=\frac{b^2-a^2-c^2}{2c}=r_1$$

$$\mathrm{OB}=\frac{b^2+c^2-a^2}{2c}=r_2$$

$$c=r_2-r_1$$

$$q_1=\frac{r_1-\lambda}{a}=\frac{a}{r_1+\lambda}$$

$$q_2=\frac{r_2-\lambda}{b}=\frac{b}{r_2+\lambda}$$

$$q=\frac{q_1}{q_2}$$

$$\rho_n=(r_1-\lambda)\frac{1-q^{2n}}{1-q_1^2q^{2n}}$$

which is the same *form* as before, only q is the inverse of its former value.

And, as before,

$$2\mathrm{T}=\tfrac{1}{2}\mathrm{M}_1u^2\left\{1+3(1-q_1^2)^3\Sigma_1^\infty\left(\frac{q^n}{1-q_1^2q^{2n}}\right)^3\right\}$$

$$=\tfrac{1}{2}\mathrm{M}_1u^2\{1+3\mathrm{Q}(q.q_1)\}$$

A table for Q when $b=2a$ is given at the end of the paper.

10. It will be well here, before passing on to the consideration of the motion, to make a short digression on the properties of the functions Q and Q′. In the first place it is easily seen that the series for the Q and Q′ functions are both convergent, even up to the case when the spheres touch, or $q=1$; for the ratio of the n^{th} term to the $n-1^{\text{th}}$ is

$$\left\{q\frac{1-q_1^{-2}q^{2n-2}}{1-q_1^{-2}q^{2n}}\right\}^3$$

and this is always less than q^3, which is less than unity, except in the case when the spheres touch. In this particular case the n^{th} term tends to the limit

$$\left(\frac{1}{1-n\dfrac{dq}{dq_1}}\right)^3$$

and the series is still convergent. The value of $\frac{dq}{dq_1}$ is the limit when $\lambda=0$ of

$$\frac{q_1-q_2}{1-q_1}=-\frac{a+b}{b}.$$

Hence when the spheres are in contact

$$Q=\Sigma_1^\infty\left(\frac{b}{n(a+b)+b}\right)^3=x^3\Sigma_1^\infty\left(\frac{1}{n+x}\right)^3=-\tfrac{1}{2}x^3\frac{d^3}{dx^3}\log\Gamma(1+x) \quad . \quad . \quad . \quad . \quad (9)$$

if $x=\frac{b}{a+b}$. The values of this may be found from LEGENDRE's table of the log Γ-functions.

If the spheres be equal $x=\frac{1}{2}$ and

$$Q=\Sigma_1\frac{1}{(2n+1)^3}=S'_3-1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now

$$S_3=1+\frac{1}{2^3}+\frac{1}{3^3}+ \ldots . =1\cdot202056903159 \ldots .$$

$$=S'_3+\tfrac{1}{8}S_3$$

whence

$$S'_3=\tfrac{7}{8}S_3=1\cdot051799790264$$

When the spheres are equal $q_2=\frac{1}{q_1}$. If in this case q denote either q_2 or $\frac{1}{q_1}$

$$Q=(1-q^2)^3\Sigma_1^\infty\left(\frac{q^{2n}}{1-q^{4n+2}}\right)^3$$

11. We may easily express the general term in terms of r, a, b. For writing it in the form

$$Q=\left\{\frac{q_1-\frac{1}{q_1}}{\frac{q_1}{q^n}-\frac{q^n}{q_1}}\right\}^3=u_n^3 \text{ suppose,}$$

we get at once from the relations (2), (3), (4)

$$u_n=\frac{2\lambda a^n b^n}{(r_1+\lambda)^{n+1}(r_2+\lambda)^n-(r_1-\lambda)^{n+1}(r_2-\lambda)^n}$$

which, since $r_2=r-r_1$ and $r_1^2-\lambda^2=a^2$

$$=\frac{2\lambda a^n b^n}{(r_1+\lambda)\{r(r_1+\lambda)-a^2\}^n-(r_1-\lambda)\{r(r_1-\lambda)-a^2\}^n}$$

Now

$$rr_1-a^2=\frac{r^2+a^2-b^2}{2}-a^2=\frac{r^2-a^2-b^2}{2}=\frac{x^2}{2} \text{ suppose,}$$

also

$$\lambda^2r^2=(r_1^{\,2}-a^2)r^2=\frac{(r^2+a^2-b^2)^2-4a^2r^2}{4}=\frac{x^4-4a^2b^2}{4}$$

We shall further write $4a^2b^2=\alpha^4$

Then

$$u_n=\frac{2\lambda a^nb^n}{r_1\left\{\left(\frac{x^2}{2}+\lambda r\right)^n-\left(\frac{x^2}{2}-\lambda r\right)^n\right\}+\lambda\left\{\left(\frac{x^2}{2}+\lambda r\right)^n+\left(\frac{x^2}{2}-\lambda r\right)^n\right\}}$$

$$=\frac{2^na^nb^n}{2rr_1\Sigma\left\{\frac{\lfloor n}{\lfloor 2p+1\,\lfloor n-2p-1}.x^{2n-4p-2}(2\lambda r)^{2p}\right\}+\Sigma\frac{\lfloor n}{\lfloor 2p\,\lfloor n-2p}x^{2n-4p}(2\lambda r)^{2p}}$$

$$=\frac{2^na^nb^n}{\Sigma\frac{\lfloor n}{\lfloor 2p+1\,\lfloor n-2p}\{(n+1)x^2+2(n-2p)a^2\}x^{2n-4p-2}(x^4-\alpha^4)^p}=\frac{2^na^nb^n}{v_n} \quad . \quad . \quad (11)$$

and

$$v_n=\Sigma\Sigma\frac{\lfloor n\,\lfloor p\,(-1)^q}{\lfloor 2p+1\,\lfloor n-2p\,\lfloor q\,\lfloor p-q}\{\overline{n+1}x^2+2(n-2p)a^2\}x^{2n-4q-2}\alpha^{4q}$$

Denote

$$\Sigma_{p=q}\frac{\lfloor n}{\lfloor 2p+1\,\lfloor n-2p-1}.\frac{\lfloor p}{\lfloor q\,\lfloor p-q}$$

by $S_{n.q}$

Then

$$v_n=\Sigma\overline{-1}|^q\{S_{n+1,q}.x^{2n-4q}+2S_{n.q}a^2x^{2n-4q-2}\}\alpha^{4q}$$

Let

$$y=\frac{(1+\sqrt{x})^n-(1-\sqrt{x})^n}{2\sqrt{x}}=n+\frac{n(n-1)(n-2)}{\lfloor 3}x+\ldots\ldots$$

$$+\frac{\lfloor n}{\lfloor 2p+1\,\lfloor n-2p-1}x^p+\ldots\ldots$$

Then

$$S_{n.0}=\text{value of } y \text{ when } x \text{ is } 1=2^{n-1}$$

$$S_{n.1}=\text{value of } \frac{dy}{dx} \text{ when } x \text{ is } 1=(n-1)2^{n-2}$$

and in general

$$S_{n.q}=\text{value of } \frac{1}{\lfloor q}\frac{d^qy}{dx^q} \text{ when } x=1$$

Now $q<n$. Hence

$$\frac{d^q}{dx^q}\frac{(1-\sqrt{x})^n}{\sqrt{x}}=0 \text{ when } x=1$$

Therefore

$$S_{n.q}=\frac{1}{2\underline{|q}}\left[\frac{d^q}{dx^q}.\frac{(1+\sqrt{x})^n}{\sqrt{x}}\right]_{x=1}$$

Also we see that v_n is a rational integral function of $(r^2-a^2-b^2)$

When one sphere is inside the other the series for Q is still convergent.

12. When the spheres are concentric

$$r_1=r_2=\infty \quad \lambda=\infty$$

$$q_1=0 \quad q_2=0$$

$$q=\frac{q_1}{q_2}=\frac{a}{b}$$

and

$$Q=.\left(\frac{a}{b}\right)^3\frac{1}{1-\left(\frac{a}{b}\right)^3}=\frac{a^3}{b^3-a^3}$$

whence

$$2T=\tfrac{1}{2}.\frac{b^3+2a^3}{b^3-a^3}M_1u^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which agrees with the result found by STOKES in his paper of 1843 before referred to.

When the inner touches the outer $\lambda=0$ and

$$Q=x^3\Sigma\frac{1}{(n+x)^3}=-\tfrac{1}{2}x^3\frac{d^3}{dx^3}\log_e\Gamma(1+x)$$

$$=-1{\cdot}15129x^3\frac{d^3}{dx^3}\log_{10}\Gamma(1+x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where

$$x=\frac{b}{b-a}$$

If x is an integer $=m$ say

$$Q=m^3\left\{S_3-\Sigma_1^m\frac{1}{n^3}\right\}=m^3\left\{{\cdot}2020569-\Sigma_2^m\frac{1}{n^3}\right\}$$

a finite expression, and in this case

$$a=\frac{m-1}{m}b$$

In the particular cases

$$a=\tfrac{1}{2}b \qquad Q=\ {\cdot}61645$$

$$a=\tfrac{2}{3}b \qquad Q=1{\cdot}08054$$

If x is of the form $\frac{2m+1}{2}$,

$$Q=(2m+1)^3\left\{S'_3-\Sigma_0^m\left(\frac{1}{2n+1}\right)^3\right\}$$

$$=(2m+1)^3\left\{\cdot 0517998-\Sigma_1^m\left(\frac{1}{2n+1}\right)^3\right\}$$

Also a finite expression, and in this case

$$a=\frac{2m-1}{2m+1}b$$

In the particular cases

$$a=\tfrac{1}{3}b \quad Q=\cdot 39859$$

$$a=\tfrac{3}{5}b \quad Q=\cdot 84535$$

The expressions for Q directly in terms of r, a, b are the same functions of $a^2+b^2-r^2$ as the corresponding expressions for external spheres are of $r^2-a^2-b^2$.

13. The series for Q′ may, as in the case of Q, be shown to be convergent.

When the spheres are in contact

$$Q'=\frac{a^3b^3}{r^3}S_3=\left(\frac{ab}{a+b}\right)^3 S_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Also the general term in r, a, b is given by

$$u_n=\frac{(2ab)^n}{2r\Sigma\frac{\lfloor n}{\lfloor 2p+1\,\lfloor n-2p-1}x^{2n-4p-2}(x^4-a^4)^p}=\frac{2^{n-1}a^nb^n}{r\Sigma(-1)^q S_{n\cdot q}a^{4q}x^{2n-4q-2}} \quad . \quad . \quad (15)$$

It is easily seen that both Q, Q′ for external spheres diminish as x—*i.e.*, r—increases.

Hence for external spheres $\frac{dQ}{dr}$, $\frac{dQ'}{dr}$, are both negative.

When one sphere is inside the other, Q decreases as x increases—*i.e.*, as r diminishes.

Hence in this case $\frac{dQ}{dr}$ is positive.

The values of the first three terms of Q, Q′ are

$$\left.\begin{array}{ll}\text{for Q} & \left\{\frac{ab}{r^2-b^2}\right\}^3, \left\{\frac{a^2b^2}{(r^2-b^2)^2-a^2r^2}\right\}^3, \left\{\frac{a^3b^3}{x^6+a^2x^4-2a^2b^2x^2-a^4b^2}\right\}^3 \\ \text{and for Q}' & \left(\frac{ab}{r}\right)^3, \left(\frac{a^2b^2}{rx^2}\right)^3, \left\{\frac{a^3b^3}{r(x^4-a^2b^2)}\right\}^3\end{array}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

14. We may easily find $\frac{dQ_1}{dr}, \frac{dQ'}{dr}$ at contact of the spheres. If Q_n denote the n^{th} term of Q_1, then it may be shown that, x denoting $\frac{b}{a+b}$

$$\frac{dQ_n}{dr} = -\frac{n(n+1)(n-1+3x)}{a(n+x)^4} x^2$$

$$\frac{dQ'_n}{a^3 dr} = -\frac{n^2-1+3x(1-x)}{an^3} x^2$$

both of which are of the order $\frac{1}{n}$. Hence the values of $\frac{dQ_1}{dr}, \frac{dQ_2}{dr}, \frac{dQ'}{dr}$ at contact are $= -\infty$. But though this is the case, the value of $\frac{d}{dr}(Q_1 - \frac{1}{a^3}Q')$ at contact is finite. The n^{th} term is

$$\frac{n^3(n+1)(n-4) + 3(1-x)(n+x)^4 + (n^2-1)(6n^2+4nx+x^3)x}{an^3(n+x)^4} x^3$$

which is of the order $\frac{1}{n^2}$, and therefore the whole sum is finite. Also when $n \overline{>} 2$ the n^{th} term is positive, when $n=1$ the sign depends on the value of x. But by considering the values of Q, &c., in terms of r, expanding them in ascending inverse powers of r, it can be shown that $\frac{d}{dr}\left(Q - \frac{1}{a^3}Q'\right)$ is positive always. Further, at contact $Q - \frac{Q'}{a^3}$ is a negative quantity, whilst at an infinite distance it is zero. Hence, *on the whole*, it must increase with r, and if this takes place continuously, $\frac{d}{dr}\left(Q - \frac{1}{a^3}Q'\right)$ would always be positive. Though I have convinced myself that such is the case, I have not been able to prove it in general. When the spheres are at a great distance the values of Q and Q' depend only on their first terms, and $Q - \frac{1}{a^3}Q'$ only on the term of Q', which is of the order $\frac{b^3}{r^3}$. Hence here also the differential coefficient is positive. I have calculated and laid down curves representing the magnitudes of the Q and Q'-functions in the case of equal spheres, and when the radius of one sphere is twice that of the other, and in both cases the value for $\frac{d}{dr}\left(Q - \frac{1}{a^3}Q'\right)$ comes out positive for all distances. In what follows we shall suppose that this quantity is always positive, but it must be understood throughout as only *proved* for the case of equal spheres and the case in which the radius of one sphere is double that of the other.

15. Although the rapidly increasing complexity of the successive images when the spheres move perpendicularly to their line of centres would lead us to regard the

problem of finding the energy in this case as almost hopeless, yet we can carry the approximation to any number of images with less labour than might at first sight appear. For suppose we wish to take into account $2n$ images in A, due to A's motion, that is on the whole $4n$ reflections. We need only first calculate the distribution of doublets for a *general* position of the original one, in the n^{th} image in A, and find the amount of the first n images. We can then treat the second portion of the $2n$ images as the images resulting from the different parts of the n^{th} image, and employ our first result to find the amount of the second portion by a single integration. Suppose we proceed as if we were going on indefinitely: we suppose an original doublet in A at a distance ρ and calculate the density of the parts of the first image in A, say $f(r)$ at a distance r, and thence its amount. We employ this result to find the density at any point of the second image, regarding it as made up of images of the different parts of the first, and this we do by using the expression found before, substituting for the original doublet at ρ, an amount $f(r)dr$ at a distance r, and integrating with respect to r over the first image. Thus we find the distribution for the second image and its amount, and therefore the amount for the first two images together. Starting now from this, and proceeding in the same way, we find the distribution and amount of the first four images, then of the first eight, and so on. Thus to find the distribution of the 2^pth image we only require $\overline{p+1}$ operations, and to find its amount only p operations. Even with this method of proceeding the work would be exceedingly laborious. But for all practical purposes the first two images in A, *i.e.*, the motion due to *four reflections*, will be sufficient—except when the spheres are in contact. We proceed then to find the values of the coefficient of v^2 and of $v_1 v_2$ to this degree of approximation.

Suppose we have at P inside A a doublet k at a distance ρ_1 from A, whose axis is perpendicular to A B.

i. *First image in* B.—Then we have at Q_1, its inverse point in B, a doublet $\left(\frac{b}{BP_1}\right)^3 k$ and a line doublet thence to B, whose line density $=-\frac{k}{b}\cdot\frac{r}{BP_1}$.

ii. *First image in* A.—The image of this in A consists of two parts, that depending on the single doublet in B, and that depending on the line doublet.

(α) *Image of* Q_1.—A doublet at P_2 $\left(AP_2=\frac{a^2}{AQ_1}\right)$ whose magnitude is $\left(\frac{ab}{AQ_1.BP_1}\right)^3 k$, and a negative line doublet from P_2 to A whose line density $=-\frac{k}{a}\cdot\left(\frac{b}{BP_1}\right)^3\frac{R}{AQ_1}$

(β) *Image of negative line doublet.*—At a distance r from B we have a negative doublet $=-\frac{k}{b}\cdot\frac{r}{BP_1}dr$. This has (1) a negative doublet at a distance from $A=\frac{a^2}{c-r}=R$ equal to $-\left(\frac{a}{c-r}\right)^3\frac{k}{b}\cdot\frac{r}{BP_1}dr$. That is from P_2 to A we have a negative line doublet whose density at a distance R is

$$-\left(\frac{a}{c-r}\right)^3 \frac{k}{b}\cdot\frac{r}{\mathrm{BP}_1}\frac{dr}{d\mathrm{R}}$$

and $c-r=\dfrac{a^2}{\mathrm{R}}$

$$\therefore \frac{dr}{d\mathrm{R}}=\frac{a^2}{\mathrm{R}^2}$$

$$\therefore \text{density} =-\frac{k}{ab}\cdot\frac{c\mathrm{R}-a^2}{\mathrm{BP}_1}$$

(2) a line doublet image of each portion. The doublet $-\frac{k}{b}\cdot\frac{r}{\mathrm{BP}_1}dr$ produces a positive line doublet from a distance $\mathrm{R}'=\frac{a^2}{c-r}$ to A, whose line density $=\frac{k}{ab}\cdot\frac{rdr}{\mathrm{BP}_1}\cdot\frac{\mathrm{R}}{c-r}$.

Hence the density at a distance R, due to this part, from the whole line doublet in B

$$=\int_{c-\frac{a^2}{\mathrm{R}}}^{\mathrm{BQ}_1}\frac{k}{ab}\cdot\frac{\mathrm{R}}{\mathrm{BP}_1}\cdot\frac{rdr}{c-r}$$

$$=\frac{k}{ab}\cdot\frac{\mathrm{R}}{\mathrm{BP}_1}\left\{c\log\frac{\rho_2}{\mathrm{R}}-a^2\left(\frac{1}{\mathrm{R}}-\frac{1}{\rho_2}\right)\right\}$$

Hence finally the density at a distance R from A of the resultant line doublet

$$=-\frac{k}{a}\cdot\left(\frac{b}{\mathrm{BP}_1}\right)^3\frac{\mathrm{R}}{\mathrm{AQ}_1}-\frac{k}{ab}\cdot\frac{c\mathrm{R}-a^2}{\mathrm{BP}_1}+\frac{k}{ab}\cdot\frac{\mathrm{R}}{\mathrm{BP}_1}\left\{c\log\frac{\rho_2}{\mathrm{R}}-a^2\left(\frac{1}{\mathrm{R}}-\frac{1}{\rho_2}\right)\right\}$$

$$=-\frac{k}{a}\cdot\frac{\mathrm{R}}{\mathrm{AQ}_1}\cdot\left(\frac{b}{\mathrm{BP}_1}\right)^3+\frac{k}{ab}\cdot\frac{\mathrm{R}}{\mathrm{BP}_1}\left\{c\log\frac{\rho_2}{\mathrm{R}}-\mathrm{BQ}_1\right\}$$

and the whole amount

$$=-\frac{k}{a}\int_0^{\rho_2}\left\{\frac{1}{\mathrm{AQ}_1}\cdot\left(\frac{b}{\mathrm{BP}_1}\right)^3+\frac{\mathrm{BQ}_1}{b.\mathrm{BP}_1}-\frac{c}{b.\mathrm{BP}_1}\log\frac{\rho_2}{\mathrm{R}}\right\}\mathrm{R}d\mathrm{R}$$

$$=-k\left\{\frac{1}{2}\left(\frac{ab}{\mathrm{AQ}_1.\mathrm{BP}_1}\right)^3+\frac{1}{2}\frac{\mathrm{BQ}_1.a^3}{b.\mathrm{BP}_1.\mathrm{AQ}_1^2}-\frac{1}{4}\frac{a^3c}{b.\mathrm{BP}_1.\mathrm{AQ}_1^2}\right\}$$

So that the whole amount of the image is

$$\frac{1}{2}\cdot\left(\frac{ab}{\mathrm{AQ}_1.\mathrm{BP}_1}\right)^3k+\frac{1}{4}\frac{a^3}{b.\mathrm{BP}_1.\mathrm{AQ}_1^2}(c-2\mathrm{BQ}_1)k$$

or substituting for AQ_1, &c., in terms of ρ_1

$$\frac{1}{2}\left(\frac{ab}{c^2-b^2-c\rho_1}\right)^3k+\frac{a^3(c^2-c\rho_1-2b^2)}{4b(c^2-b^2-c\rho_1)^2}k$$

For the first image in A, $\rho_1=0$

Hence

$$\mu_1=\tfrac{1}{2}\left\{\left(\frac{ab}{c^2-b^2}\right)^3+\frac{a^3(c^2-2b^2)}{2b(c^2-b^2)^2}\right\}k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The density at any point of the first image is

$$-\frac{kb}{a}\left\{\frac{1}{c^2-b^2}-\frac{1}{b^2}\log\frac{\rho_2}{R}\right\}R$$

together with a doublet $\left(\frac{ab}{c^2-b^2}\right)^3 k$ at a distance ρ_2

The amount of that part of the second image in A which depends on the latter is

$$\tfrac{1}{2}\left\{\left(\frac{ab}{c^2-b^2-c\rho_2}\right)^3+\frac{a^3(c^2-c\rho_2-2b^2)}{2b(c^2-b^2-c\rho_2)^2}\right\}\left(\frac{ab}{c^2-b^2}\right)^3 k$$

and the amount of the part due to the portion of the former at a distance R is

$$-\frac{kb}{2a}\left\{\left(\frac{ab}{c^2-b^2-cR}\right)^3+\frac{a^3(c^2-2b^2-cR)}{2b(c^2-b^2-cR)^2}\right\}\left\{\frac{1}{c^2-b^2}-\frac{1}{b^2}\log\frac{\rho_2}{R}\right\}RdR$$

whence the whole amount due to the former

$$=-\frac{kb}{2ac}\int_0^{\rho_2}\left[\frac{a^3b^3(c^2-b^2)}{(c^2-b^2-cR)^3}-\frac{a^3b(c^2+b^2)}{2(c^2-b^2-cR)^2}+\frac{a^3c^2}{2b(c^2-b^2-cR)}-\frac{a^3}{2b}\right]\left\{\frac{1}{c^2-b^2}-\frac{1}{b^2}\log\frac{\rho_2}{R}\right\}dR$$

$$=-\frac{ka^2b}{4c^2(c^2-b^2)}\left[\frac{b^3(c^2-b^2)}{(c^2-b^2-cR)^2}-\frac{b(c^2+b^2)}{c^2-b^2-cR}-\frac{cR}{b}-\frac{c^2}{b}\log(c^2-b^2-cR)\right]_0^{\rho_2}$$

$$+\frac{ka^2}{4b^2}\int_0^{\rho_2}c\frac{\log\frac{\rho_2}{R}}{c^2-b^2-cR}dR$$

$$+\frac{ka^2}{4bc^2}\int_0^{\rho_2}\log\frac{\rho_2}{R}\,\frac{d}{dR}\left\{\frac{b^3(c^2-b^2)}{(c^2-b^2-cR)^2}-\frac{b(c^2+b^2)}{c^2-b^2-cR}-\frac{cR}{b}+\frac{bc^2}{c^2-b^2}\right\}dR$$

$$=-\frac{ka^2b^4}{4c^2(c^2-b^2-c\rho_2)^2}+\frac{ka^2}{4(c^2-b^2)}\left\{\frac{2b^2}{c^2-b^2-c\rho_2}+\frac{2b^2-c^2}{b^2.c}\rho_2+\frac{b^2(b^2-2c^2)}{c^2(c^2-b^2)}\right.$$

$$\left.+2\log\frac{c^2-b^2-c\rho_2}{c^2-b^2}\right\}-\frac{ka^2}{4b^2}\int_0^1\frac{\log x}{\frac{c^2-b^2}{c\rho_2}-x}dx$$

Now $c\rho_2=\frac{a^2c^2}{c^2-b^2}$, $=ac\alpha$ say.

Then the above is

$$=-\frac{kb^4}{4c^4}\left(\frac{\alpha}{1-\alpha^2}\right)^2+\frac{ka^2}{4c^2}\left\{\frac{2b^2}{1-\alpha^2}+2(a^2-b^2)-\frac{a^2c^2}{b^2}+\frac{b^4}{c^2}+2(c^2-b^2)\log(1-\alpha^2)\right\}-\frac{ka^2}{4b^2}\int_0^{\alpha^2}\frac{\log x}{\frac{1}{\alpha^2}-x}dx$$

Wherefore the whole amount of the second image in A_1 is $\left(\text{writing } \beta=\frac{ba}{c^2-b^2}\right)$

$$\frac{\mu_2}{\mu_0}=\tfrac{1}{2}\left(\frac{\beta^2}{1-\alpha^2}\right)^3-\tfrac{1}{4}\left(\frac{a\beta^3}{b}+\frac{b^2}{c^2}\right)\left(\frac{\beta}{1-\alpha^2}\right)^2+\tfrac{1}{4}\left(\frac{a^2\beta^2}{b^2}+2\right)\frac{\beta^2}{1-\alpha^2}$$
$$+\tfrac{1}{2}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}\right)\beta^2-\tfrac{1}{2}\beta^2+\frac{a^2\alpha^2}{2c^2}\log(1-\alpha^2)-\frac{a^2}{4b^2}\int_0^{\alpha^2}\frac{\log x}{\frac{1}{\alpha^2}-x}dx\quad . \quad . \quad . \quad . \quad (18)$$

Substituting for $\frac{\mu_2}{\mu_0}$ and $\frac{\mu_1}{\mu_0}$ in (7) we get the part of T depending on v_1^2 correct to the second image in A. Interchanging a and b, the part depending on v_2^2 is found.

In the case of equal spheres

$$\frac{\mu_1}{\mu_0}=\tfrac{1}{2}\left\{\left(\frac{a^2}{c^2-a^2}\right)^3+\frac{a^2(c^2-2a^2)}{2(c^2-a^2)^2}\right\}$$

$$\frac{\mu_2}{\mu_0}=\tfrac{1}{2}P^3-\tfrac{1}{4}\cdot\frac{a^4c^2+(c^2-a^2)^3}{b^2c^2(c^2-a^2)}P^2+\tfrac{1}{4}\left(\frac{a^4}{(c^2-a^2)^2}+2\right)P$$
$$+\frac{a^6}{(c^2-a^2)^2c^2}+\frac{a^4}{2(c^2-a^2)^2}\log(1-\alpha^2)-\tfrac{1}{4}\int_0^{\alpha^2}\frac{\log x}{\frac{1}{\alpha^2}-x}dx$$

16. To find the value of the coefficient of the term in $v_1\, v_2$ we need to find the amounts of the images in B due to the motion of A, and *vice versâ*.

The first image in B of k in A at a distance ρ_1 is $\left(\frac{b}{BP_1}\right)^3 k$ at Q_1 and a negative line doublet thence to B, whose line density is $-\frac{k}{b}\cdot\frac{r}{BP_1}$.

The whole amount is therefore

$$\left(\frac{b}{BP_1}\right)^3 k-\tfrac{1}{2}\left(\frac{b}{BP_1}\right)^3 k=\tfrac{1}{2}\left(\frac{b}{c-\rho_1}\right)^3 k$$

For the first $\rho_1=0$ and $\frac{\nu_1}{\mu_0}=\tfrac{1}{2}\left(\frac{b}{c}\right)^3 k$

To find the amount of the second image in B we start from the first image in A already found. This is, as has been shown, a doublet at $\rho_2=\left(\frac{ab}{c^2-b^2}\right)^3 k$ and a line doublet thence to A, whose line density is

$$\left\{\frac{c}{b.BP_1}\left(\log\frac{\rho_2}{R}-\frac{BQ_1}{c}\right)-\frac{1}{AQ_1}\left(\frac{b}{BP_1}\right)^3\right\}\frac{kR}{a}$$

The amount in B from the former is

$$\tfrac{1}{2}\left\{\frac{ab^2}{(c-\rho_2)(c^2-b^2)}\right\}^3 k$$

and from the second is

$$\tfrac{1}{2}\int_0^{\rho_2}\left(\frac{b}{c-\mathrm{R}}\right)^3\left\{\frac{1}{b}\log\frac{\rho_2}{\mathrm{R}}-\frac{b}{c^2-b^2}\right\}\frac{k\mathrm{R}}{a}d\mathrm{R}$$

$$=\frac{kb^2}{4ac}\left\{\frac{\rho_2}{c-\rho_2}+\tfrac{1}{2}\log\frac{c-\rho_2}{c}-\frac{b^2\rho_2^2}{(c^2-b^2)(c-\rho_2)^2}\right\}$$

$$=\frac{kb^2}{4ac}\left\{\frac{c\rho_2}{c^2-b^2}\cdot\frac{c^2-b^2-c\rho_2}{(c-\rho_2)^2}+\tfrac{1}{2}\log\frac{c-\rho_2}{c}\right\}$$

$$\therefore=\frac{kb^2}{4ac}\left\{\frac{a^2}{c^2-b^2}\cdot\frac{(c^2-b^2)^2-a^2c^2}{(c^2-a^2-b^2)^2}+\tfrac{1}{2}\log\left(.1-\frac{a^2}{c^2-b^2}\right)\right\}$$

$$\therefore\frac{\nu_2}{\mu_0}=\tfrac{1}{2}\left\{\frac{ab^2}{c(c^2-a^2-b^2)}\right\}^3+\frac{b^2}{4ac}\{\;.\;.\;.\;.\;\}$$

So also

$$\frac{\mu'_2}{\nu_0}=\tfrac{1}{2}\left\{\frac{a^2b}{c(c^2-a^2-b^2)}\right\}^3+\frac{a^2}{4bc}\left\{\frac{b^2}{c^2-a^2}\cdot\frac{(c^2-a^2)^2-b^2c^2}{(c^2-a^2-b^2)^2}+\tfrac{1}{2}\log\frac{c^2-a^2-b^2}{c^2-a^2}\right\}$$

Whence from (8)

$$\mathrm{L}'=2\pi b^3\left(\frac{\mu'_1+\mu'_2}{\nu_0}\right)+2\pi a^3\left(\frac{\nu_1+\nu_2}{\mu_0}\right)$$

$$=\tfrac{3}{2}\mathrm{M}'\left[\left(\frac{ab}{c}\right)^3+\left\{\frac{a^2b^2}{c(c^2-a^2-b^2)^2}\right\}^3-\frac{a^2b^2}{8c}\log\frac{(c^2-a^2-b^2)^2}{(c^2-a^2)(c^2-b^2)}\right.$$

$$\left.+\frac{a^2b^2}{4c(c^2-a^2-b^2)^2}\left\{c^2(a^2+b^2)-2a^2b^2-c^2\left(\frac{b^4}{c^2-a^2}+\frac{a^4}{c^2-b^2}\right)\right\}\right]\quad . \quad . \quad (19)$$

Similarly can be found the coefficient of v^2 when one sphere moves inside another.

Motion in the line of centres.

17. When the two spheres are moving in the line of centres the kinetic energy is given by

$$2\mathrm{T}=\mathrm{A}_1u_1^2+\mathrm{A}_2u_2^2-2\mathrm{B}u_1u_2$$

where

$$\mathrm{A}_1=m_1+\tfrac{1}{2}m'_1\left\{1+3\mathrm{Q}\left(\frac{1}{q_1}.q\right)\right\}$$

$$\mathrm{B}=\tfrac{3}{2}\mathrm{M}'\mathrm{Q}'(q)$$

and m_1, m_1', M′ respectively denote the mass of the sphere (A), the mass of fluid displaced by it, and the mass of fluid in a unit sphere.

It is to be remarked that A_1, A_2, B are functions only of the distance between the spheres, and that therefore $\frac{d}{dx_1}+\frac{d}{dx_2}=0$. Since no forces are supposed to act on the system, both the energy and momentum are constant. Hence

$$\left.\begin{aligned} &2T=\text{constant}\\ &\frac{\delta T}{\delta u_1}+\frac{\delta T}{\delta u_2}=\text{constant}=d \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The last equation also follows at once from LAGRANGE'S equation since $\frac{\delta T}{\delta x_1}+\frac{\delta T}{\delta x_2}=0$, and may be written

$$(A_1-B)u_1+(A_2-B)u_2=d$$

We shall transform these equations by referring the motion to the velocity of an arbitrarily chosen point P between the spheres, and the distance between them.

Let P divide the distance (r) in the constant ratio $\frac{\alpha}{1-\alpha}=\frac{\alpha}{\beta}$. Then if x is the distance of P from the origin, u its velocity

$$x_1=x+\alpha r, \quad x_2=x-\beta r$$

and

$$u_1=u+\alpha\dot{r}, \quad u_2=u-\beta\dot{r}$$

whence

$$\left.\begin{aligned} &(A_1+A_2-2B)u^2+(A_1\alpha^2+A_2\beta^2+2\alpha\beta B)\dot{r}^2\\ &\qquad\qquad +2\{\alpha(A_1-B)-\beta(A_2-B)\}u\dot{r}=2T\\ &(A_1+A_2-2B)u+\{\alpha(A_1-B)-\beta(A_2-B)\}\dot{r}=d \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21)$$

which we shall write

$$\left.\begin{aligned} &pu^2+q\dot{r}^2+2lu\dot{r}=2T\\ &pu+l\dot{r}=d \end{aligned}\right\}$$

whence

$$(pq-l^2)\dot{r}^2=2Tp-d^2$$

or

$$\dot{r}=\pm\sqrt{\left(\frac{2Tp-d^2}{A_1A_2-B^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

in which we are to take the positive or negative sign according as the spheres are separating or approaching one another. The spheres will move as if they repel or attract one another *relatively* according as $\frac{d}{dr}\left\{\frac{2\mathrm{T}p-d^2}{\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2}\right\}$ is positive or negative. This condition does not depend on their relative motion at any time, but only on their distance and the ratio of the constant energy to the constant momentum. The above condition may also be expressed, writing $\frac{d^2}{2\mathrm{T}}=k^2$, as the sign of

$$k^2\frac{d}{dr}(\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2)-\left\{(\mathrm{A}_2-\mathrm{B})^2\frac{d\mathrm{A}_1}{dr}+(\mathrm{A}_1-\mathrm{B})^2\frac{d\mathrm{A}_2}{dr}+2(\mathrm{A}_1-\mathrm{B})(\mathrm{A}_2-\mathrm{B})\frac{d\mathrm{B}}{dr}\right\}$$

The last term is positive, for A_1, A_2, B all decrease as r increases. Now k must always be $<p$ since $\dot{r}$ is always real. If we put $k^2=p=\mathrm{A}_1+\mathrm{A}_2-2\mathrm{B}$ in the above, the criterion reduces to the sign of

$$(\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2)\frac{d}{dr}\{\mathrm{A}_1+\mathrm{A}_2-2\mathrm{B}\}$$

i.e., since $\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2$ is always positive to the sign of

$$\frac{d}{dr}(\mathrm{A}_1-\mathrm{B})+\frac{d}{dr}(\mathrm{A}_2-\mathrm{B})$$

Now we are led to conclude from the argument in § 14 that $\frac{d}{dr}(\mathrm{A}_1-\mathrm{B})$ . . . are always positive. Hence when k has its greatest possible value the criterion is positive, much more then is it so for any other value of k. Hence we are led to conclude that whatever be the relation between the momentum and energy the spheres always move so that $\dot{r}$ tends to decrease, whilst in the case of equal spheres, or that in which the radius of one is twice that of the other, we know for certain that such is the case. We cannot prove from this that the spheres move with reference to a *fixed* point as if they repel one another, for it might happen that both the spheres might be accelerated, the extra energy of the motion of the spheres themselves being taken from the fluid motion; or that both are even retarded. We can easily show, however, that both cannot be accelerated if $\dot{r}$ is positive and both move in the same direction, for the distance in this case increases, and therefore so do $\mathrm{A}_1-\mathrm{B}$, $\mathrm{A}_2-\mathrm{B}$, and hence because $(\mathrm{A}_1-\mathrm{B})u_1+(\mathrm{A}_2-\mathrm{B})u_2$ is constant u_1, u_2 cannot both increase. Also if $\dot{r}$ is negative and u_1, u_2 of the same sign the same result holds.

In the case where the spheres are projected so that the momentum is zero

$$\dot{r}^2=\frac{2\mathrm{T}p}{\mathrm{A}_1\mathrm{A}_2-\mathrm{B}^2}$$

and the relation between the velocities of projection that this may be the case is given by

$$\frac{u_1}{u_2}=-\frac{A_2-B}{A_1-B}$$

When the spheres are equal $u_2=-u_1$ and the motion is the same as that of a single sphere in a fluid bounded by a plane, and moving perpendicularly to the plane.

For this particular case

$$\dot{r}^2=\frac{4T}{A+B}$$

or if u denote the velocity relative to the fixed plane $\dot{r}=2u$, and

$$u^2=\frac{T}{A+B}=\frac{(A+B)_0}{A+B}u_0^2$$

where $(A+B)_0$, u_0 are the values of $A+B$, and u at the point of projection. If the sphere is projected from contact with the plane

$$\begin{aligned}(A+B)_0&=m+\tfrac{1}{2}m'+\tfrac{3}{2}m'(\tfrac{7}{8}S_3-1+\tfrac{1}{8}S_3)\\&=m+\tfrac{1}{2}m'+{\cdot}3030853m'\\&=m+{\cdot}803085m'\end{aligned}$$

and at an infinite distance

$$A+B=m+\tfrac{1}{2}m'$$

Hence the ratio of the limiting velocity to the initial velocity is

$$\sqrt{\left\{1+{\cdot}6061707\frac{1}{2\rho+1}\right\}}$$

where ρ is the density of the sphere.

For densities 0, 1, 10, the values of this ratio are respectively 1·2661, 1·0963, 1·0143. The greatest value is when the density of the sphere is zero, and the least is when $m'=0$ (no fluid) or $m=\infty$, the ratio then being, as it ought to be, unity.

In the case where the spheres are unequal and projected with no momentum from contact their initial velocities must be opposite and in the ratio of the quantities

$$m_2+\tfrac{1}{2}m'_2-\tfrac{3}{2}m'_2y^3\{\tfrac{1}{2}D^3\log_e\Gamma(1+y)+S_3\}$$

and

$$m_1+\tfrac{1}{2}m'_1-\tfrac{3}{2}m'_1x^3\{\tfrac{1}{2}D^3\log_e\Gamma(1+x)+S_3\}$$

x and y denoting the quantities $\frac{b}{a+b}$, $\frac{a}{a+b}$

If $a=2b$, $x=\frac{1}{3}$, $y=\frac{2}{3}$, and we find from LEGENDRE's tables of the Eulerian integrals

$$D^3 \log_{10} \Gamma(1+x) = -\cdot 485$$

$$D^3 \log_{10} \Gamma(1+y) = -\cdot 275$$

and the ratio is

$$\frac{1}{8}\frac{\rho+\cdot 1174}{\rho+\cdot 4642}$$

which when the densities of the spheres and fluid are equal becomes

$$\frac{\cdot 763}{8} = \cdot 0954$$

We find the velocities of the spheres relatively to the fluid by eliminating u between

$$u_1 = u + a\dot{r}$$

and

$$pu + l\dot{r} = d$$

whence

$$u_1 = \frac{d}{p} + \frac{A_2 - B}{p}\dot{r}$$

and

$$u_2 = \frac{d}{p} - \frac{A_1 - B}{p}\dot{r}$$

Suppose now the same spheres projected with the same initial circumstances except that now the spheres have changed places, and let u'_2, u'_1 be the corresponding velocities at the same distances. Then

$$u'_2 = \frac{d}{p} + \frac{A_1 - B}{p}\dot{r}$$

since d and r do not depend on the question which of the two is foremost.

Now if $a > b$ we see at once from the expressions given for A_1, A_2 in terms of the distances that $A_1 > A_2$, and hence that the foremost will be most accelerated when it is the smallest.

If now u_1, u_2 denote the velocities at any moment which we may regard as the velocity of projection

$$k^2 = \frac{d^2}{2T} = \frac{\{(A_1 - B)u_1 + (A_2 - B)u_2\}^2}{A_1 u_1^2 + A_2 u_2^2 - 2Bu_1 u_2}$$

Writing ξ for the ratio $\frac{u_1}{u_2}$ the equation to find ξ, in order that k may have a given value, is

$$\xi^2+2\frac{(A_1-B)(A_2-B)+k^2B}{(A_1-B)^2-k^2A}\xi+\frac{(A_2-B)^2-k^2A_2}{(A_1-B)^2-k^2A_1}=0$$

This enables us to find within what limits k must lie, for ξ must have real roots, and therefore

$$\{(A_1-B)(A_2-B)+k^2B\}^2-\{(A_1-B)^2-k^2A_1\}\{(A_2-B)^2-k^2A_2\}>0$$

or

$$k^2(A_1A_2-B^2)(p-k^2)>0$$

Hence k^2 may be any positive quantity less than p. The greatest possible value of this is when the spheres are infinitely distant, and then

$$p=m_1+m_2+\tfrac{1}{2}(m'_1+m'_2)$$

To each value of ξ will correspond two states of motion, the initial velocities in each case being opposite. For example, if ξ is positive, *i.e.*, both velocities in the same direction, the two states will be when (a) is the foremost, and when (b) is the foremost; if ξ be negative, the two states will be, one in which the balls begin to move towards each other, the other in which they begin to move from each other. Thus for every given value of k there are four possible states of motion.

If ever $u_1=0$ then $\xi=0$, and the spheres must be at such a distance that

$$(A_2-B)^2-k^2A_2=0$$

Now, supposing k given, this can only happen if k^2 lies between the greatest and least values of $\frac{(A_2-B)^2}{A_2}$. The least value is when the spheres are in contact, the greatest when they are at an infinite distance, the value then being $m_2+\frac{1}{2}m'_2$.

If $u_2=0$, then k^2 must lie between the greatest and least values of $\frac{(A_1-B)^2}{A_1}$. Now

$$\frac{(A_2-B)^2}{A_2}\gtrless\frac{(A_1-B)^2}{A_1}$$

as

$$(A_1A_2-B^2)(A_2-A_1)\gtrless 0$$

as

$$A_2\gtrless A_1$$

If we suppose $a>b$ then $A_1>A_2$, and calling k_1^2, k_2^2 the least values of the above limits $k_1<k$.

Hence if

$$\sqrt{\frac{d^2}{2T}}<k_1 \text{ or } >m_1+\tfrac{1}{2}m'_1$$

the spheres can neither ever come to rest; if

$$\sqrt{\frac{d^2}{2T}} < k_2 \text{ or } > m_2 + \tfrac{1}{2}m'_2$$

the small sphere can never come to rest.

The effect of the fluid on vibratory motions.

18. Suppose each of the two spheres attracted to a fixed centre of force where the force varies as the distance. Let x_1, x_2 be the distances of the spheres at any time from their respective centres of force measured in the same direction. Then

$$2T = A_1u_1^2 + A_2u_2^2 - 2Bu_1u_2 = C - m_1\mu_1x_1^2 - m_2\mu_2x_2^2$$

Also since we neglect squares of small quantities in finding the small vibrations, the equations of motion become

$$A_1\ddot{x}_1 - B\ddot{x}_2 = -m_1\mu_1x_1$$
$$-B\ddot{x}_1 + A_2\ddot{x}_2 = -m_2\mu_2x_2$$

and we suppose the spheres so distant, and their motions so small, that we may neglect the small changes in A, B during the motion. The spheres must not be too close, for at contact $\frac{dA_1}{dr}$, &c., are infinite, as was shown in § 14.

Solving the above equations in the usual manner we find

$$x_1 = L_1 \sin(K_1t + \alpha) + N_1 \sin(K_2t + \beta)$$
$$x_2 = eL_1 \sin(K_1t + \alpha) + e'N_1 \sin(K_2t + \beta)$$

where

$$\left.\begin{matrix} K_1^2 \\ K_2^2 \end{matrix}\right| = \frac{A_1m_2\mu_2 + A_2\mu_1m_1 \pm \sqrt{\{(A_1m_2\mu_2 - A_2m_1\mu_1)^2 + 4m_1m_2\mu_1\mu_2B^2\}}}{2(A_1A_2 - B^2)}$$

$$e = \frac{A_1K_1^2 - m_1\mu_1}{BK_1^2} = \frac{BK_1^2}{A_2K_1^2 - m_2\mu_2} \qquad e' = \frac{A_1K_2^2 - m_1\mu_1}{BK_2^2} = \frac{BK_2^2}{A_2K_2^2 - m_2\mu_2}$$

From this we see that, to the first order of small quantities, the mean position of the spheres is not altered, or to that degree of approximation there is no mean attraction or repulsion.

If we regard the spheres as two pendulums swinging in the fluid, in the same horizontal line, of lengths l_1, l_2, then the motion is given by the above equations if we write

$$\mu_1 = \frac{\rho_1 - 1}{\rho_1} \frac{g}{l_1} \quad \mu_2 = \frac{\rho_2 - 1}{\rho_2} \cdot \frac{g}{l_2}$$

where ρ_1, ρ_2 are the densities of the spheres compared to the fluid.

If in the above we make $m_1 = \infty$ we get the case of a forced vibration of period $\frac{2\pi}{\sqrt{\mu_1}}$ In this case

$$N_1 = 0 \quad K_1^2 = \mu_1 \quad K_2^2 = \frac{m_2 \mu_2}{A_2}$$

$$x_1 = L \sin(\sqrt{\mu_1} t + \alpha)$$

$$x_2 = \frac{B\mu_1}{A_2\mu_1 - m_2\mu_2} L \sin(\sqrt{\mu_1} t + \alpha) + N \sin\left(\sqrt{\frac{m_2\mu_2}{A_2}} t + \beta\right)$$

If the sphere (b) is set free when (a) is for the moment at rest, and the time be reckoned from this moment

$$x_2 = eL\left(\cos\sqrt{\mu_1} t - \cos\sqrt{\frac{\mu_2 m_2}{A_2}} t\right)$$

and the motion of (b) consists of two periodic terms whose amplitude is e times that of (a).

Let now the strength of the centre of force on (b) diminish indefinitely. Then

$$x_2 = \frac{B}{A_2} L(\cos\sqrt{\mu_1} t - 1)$$

and (b) would oscillate in the same period as (a), without being attracted or repelled towards it except by forces depending on the square of the amplitude of (a). To find, then, whether the action of (a) on (b) is attractive or repulsive we must take account of quantities of the second order of small quantities.

The full equation of motion of (b) is

$$A_2 \dot{u}_2 - B\dot{u}_1 - (\tfrac{1}{2}u_2^2 - u_1 u_2)\frac{dA_2}{dr} + \tfrac{1}{2}\frac{d}{dr}(A_1 - 2B)u_1^2 = 0$$

For a first approximation we have

$$x_2 = \frac{B}{A_2} L(\cos\sqrt{\mu_1} t - 1)$$

$$u_2 = -\frac{BL}{A_2}\sqrt{\mu_1} \sin\sqrt{\mu_1} t$$

Write

$$x_2 = \frac{B}{A_2} L(\cos\sqrt{\mu_1} t - 1) + z$$

where z is of the order L^2 at least. Substituting for z and neglecting cubes and higher powers of L,

$$A_2\ddot{z}-\frac{dA_2}{dr}dr.\frac{BL\mu_1}{A_2}\cos\sqrt{\mu_1}t+\frac{dB}{dr}dr L\mu_1\cos\sqrt{\mu_1}t$$

$$-\tfrac{1}{2}L^2\mu_1\left(\frac{B^2}{A_2{}^2}-2\frac{B}{A_2}\right)\frac{dA_2}{dr}\sin^2\sqrt{\mu_1}t.+\tfrac{1}{2}L^2\mu_1\frac{d}{dr}(A_1-2B)\sin^2\sqrt{\mu_1}t=0$$

and

$$dr=x_1-x_2=L\left(1-\frac{B}{A_2}\right)\cos\sqrt{\mu_1}t+\frac{BL}{A_2}$$

Whence the equation takes the form

$$A_2\ddot{z}=f+g\cos\sqrt{\mu_1}t+h\cos 2\sqrt{\mu_1}t$$

where

$$\frac{2f}{L^2\mu_1}=\left(1-\frac{B}{A_2}\right)\left(\frac{B}{A_2}\frac{dA_2}{dr}-\frac{dB}{dr}\right)+\frac{1}{2}\frac{B}{A_2}\left(\frac{B}{A_2}-2\right)\frac{dA_2}{dr}-\frac{1}{2}\frac{d}{dr}(A_1-2B)$$

$$=-\frac{1}{2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right)$$

in which last form we may neglect in A_1 the $m_1+\frac{1}{2}m'_1$ as it disappears in the differentiation. Hence the mean action of (a) on (b) is an acceleration towards (a)

$$=-\frac{L^2\mu_1}{4A_2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right)$$

$$=-\left(\frac{\pi L}{T}\right)^2\frac{1}{A_2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right)$$

$$=-\frac{v^2}{A_2}\frac{d}{dr}\left(\frac{A_1A_2-B^2}{A_2}\right)$$

if v is the "velocity of mean square" of (a).

If the distance of the spheres is so large that we may neglect twelfth and higher inverse powers of the distance, we need only consider the *first images* or the first terms in A and B. In this case it will be found that the acceleration to (a) is

$$=\frac{18v^2}{2\rho+1}\left(\frac{a}{r}\right)^6\left\{\frac{r^7}{(r^2-b^2)^4}-\frac{3}{2\rho+1}\cdot\frac{1}{r}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

To find when there is repulsion

$$\frac{r^8}{(r^2-b^2)^4}<\frac{3}{2\rho+1}$$

or

$$r > \frac{b}{\sqrt{\left\{1-\sqrt[4]{\frac{2\rho+1}{3}}\right\}}}$$

which can clearly only happen if $2\rho+1<3$ or the density of the sphere less than the fluid.

In general, then, when the body is denser than the fluid it is attracted. If its density is less than the fluid there will be a critical point (as mentioned by Sir W. THOMSON), beyond which there will be repulsion, and within which it is attractive. This critical distance is given by

$$r = \frac{b}{\sqrt{\left\{1-\sqrt[4]{\frac{2\rho+1}{3}}\right\}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

in using which it must be remembered that if r comes out nearly equal to b, the formula fails to give a correct value, as it was obtained on the supposition that the distances were large. It is, however, extremely accurate if we remember that it is true up to inverse powers of the twelfth at least. If the density of the sphere be ·9 the critical distance would be 7·648 times its radius. It may be noticed that while the principal term in the acceleration depends on r^{-7}, if the density be the same as the fluid it depends on r^{-9}.

In the case of a sphere vibrating within another sphere, along the line of centres, the effect of the fluid will be represented by supposing the inertia of the sphere increased by a mass

$$=\tfrac{1}{2}\{1+3Q(q.q_1)\times \text{ mass of fluid displaced by it}$$

where Q has the value given in § 9: provided it is not close to the boundary of the containing sphere, as in that case $\frac{dQ}{dr}$ becomes infinite, and the small motions of the sphere will produce great changes in the value of Q. When its mean position is the centre, $\frac{dQ}{dr}=0$ and Q may be considered constant when we neglect in our equations of motion cubes of small quantities. The value of Q in this case is, as has been already mentioned,

$$\tfrac{1}{2}\cdot\frac{b^3+2a^3}{b^3-a^3}\times \text{ mass of fluid displaced}$$

The foregoing serves to solve the problem of a ball pendulum within a spherical envelope when it is so suspended that its centre lies in the horizontal line through the centre of the envelope. When it oscillates in any other position the value of the coefficient of inertia may be approximated to as in §§ 15, 16.

19. If instead of supposing the sphere (b) free to move we suppose it held fast, and require to find the force necessary to do so, we get a different result from the foregoing. Suppose the sphere (a) moving in any manner, the sphere (b) being for the moment at rest, and suppose a constant force F acting on b.

The equation of motion for B is

$$A\ddot{x}_2-B\ddot{x}_1+\tfrac{1}{2}u_2(2u_1-u_2)\frac{dA_2}{dr}+\tfrac{1}{2}u_1^2\frac{d}{dr}(A_1-2B)=F$$

Suppose now that F is of such a magnitude that u_2 being zero it makes $\ddot{x}_2$ also zero. Then F is the force required to keep (b) at rest at the moment when the motion of (a) is given by u_1, $\ddot{x}_1$. Hence

$$F=-B\ddot{x}_1+\tfrac{1}{2}u_1^2\frac{d}{dr}(A_1-2B)$$

Let $x_1=L\sin Kt$, L being small. Then neglecting cubes of small quantities

$$F=\left(B+\frac{dB}{dr}dr\right)LK^2\sin Kt+\tfrac{1}{2}\frac{d}{dr}(A_1-2B)L^2K^2\cos^2 Kt$$

and

$$dr=x_1=L\sin Kt$$

$$\therefore\ F=BLK^2\sin Kt+\tfrac{1}{2}L^2K^2\left\{\frac{dB}{dr}+\tfrac{1}{2}\frac{d}{dr}(A_1-2B)\right\}+\lambda\cos 2Kt$$

This is the force at the time t necessary to keep (b) at rest. Hence the mean force is a force $=\frac{1}{4}L^2K^2\frac{dA_1}{dr}$ towards (a), which is equal and opposite to the force of (a) on (b). Since $\frac{dA_1}{dr}$ is negative, the action is an attractive one

$$=-\tfrac{1}{4}L^2K^2\frac{dA_1}{dr}$$

$$=-v^2\frac{dA_1}{dr}$$

Taking for A_1 only the first term of Q, which is equivalent to neglecting twelfth and higher inverse powers of r

$$A_1=m_1+\tfrac{1}{2}m'_1\left\{1+3\left(\frac{ab}{r^2-b^2}\right)^3\right\}$$

and the force

$$=9m'_1v^2.\frac{a^3b^3r}{(r^2-b^2)^4}$$

$$=9\frac{v^2}{g}\cdot\frac{a^3b^3r}{(r^2-b^2)^4}\times \text{weight of fluid displaced by } a \quad . \quad . \quad . \quad . \quad . \quad (25)$$

For example, for equal spheres at a distance $4a$ (distance between their surfaces $=2a$), the mean square of velocity of (a) being the same as for oxygen at a temperature of 0° C., $v=1524$ feet per 1″, and the force $=\frac{51{\cdot}6}{a}\times$ weight of fluid displaced, a being measured in feet. It is clear that while the force decreases indefinitely with the size, the "effective" force increases indefinitely.

If (a) vibrate through a distance $\frac{1}{10}$-inch, 256 times a second, and $a=\frac{1}{4}$-inch, the force is ·01197 weight of water displaced = weight of 12·8 milligrammes.

In the same manner can be found the action of (a) on (b) when a describes any small curve whose plane contains (b).

VALUES of Q, Q′ for equal spheres.

$\frac{r}{2a}$	q_2.	Q.	$\frac{1}{a^3}$ Q′.
1	1	·051800	·150257
1·05	·72985	·028307	·116749
1·1	·6418	·018768	·098312
1·2	·5367	·009531	·073754
1·35	·4431	·004049	·0511
1·5	·3819	·001959	·037142
1·75	·3138	·0007023	·023335
2	·2679	·0002962	·015631
2·5	·2087	·0000723	·008001
3·5	·1459	·0000090	·002895
4·5	·1125	·0000019	·001373

VALUES of Q_1, Q_2, Q′ when $a=2b$ for external spheres

$\frac{r}{a+b}$	Q_1.	Q_2.	$\frac{1}{a^3}$ Q′.	$\frac{1}{b^3}$ Q′.
1	·0206	·0945	·04452	·35616
1·05	·01228	·04298	·03393	·27144
1·1	·00862	·02572	·02884	·23072
1·25	·003653	·00886	·01918	·15344
1·5	·000719	·00119	·01090	·08720
2	·000186	·00024	·00455	·03643
3	..	..	·000016	·00013

VALUES of Q when $b=2a$ for an internal sphere.

$\frac{r}{a}$.	Q.
0	$\cdot\dot{1}4287\dot{0}$
·25	·15106
·5	·18046
·75	·25676
1	·61645